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Four Numerical Approaches for Solving the Radiative Transfer Equation in Magnetized White-Dwarf Atmospheres

Stefan Jordan

Institut für Astronomie und Astrophysik, University of Tübingen, Sand 1, D-72070 Tübingen, Germany

Holger Schmidt

Institut für Theoretische Physik und Astrophysik, University of Kiel, D-24098 Kiel, Germany

Abstract. We compare four different methods to calculate radiative transfer through a magnetized stellar atmosphere, and apply them to the case of magnetic white dwarfs. All methods are numerically stable enough to allow determination of the magnetic field structure, but distinctions between faster, simplifying, methods, and elaborate, but more CPU-time consuming, methods, can be made.

About 3% of all known white dwarfs have strong magnetic fields between 10^6 and 10^9 Gauss (Wickramasinghe & Ferrario 2000, Jordan 2001). The detailed surface structure of the magnetic field can be inferred from time-resolved spectro-polarimetric observations, which give integrals of the polarized radiation over the stellar surface visible at a given rotational phase. Because the magnetic field varies in strength and direction, theoretical calculations for polarized radiative transfer have to be performed at many (typically 1000) different points on the surface. In order to ensure numerical accuracy and efficiency, we have tested four different methods from the literature for solving the radiative transfer equations in magnetized white dwarf atmospheres. Note that the ALI (Accelerated Λ-Iteration) method is described in more detail by Deetjen et al. in these proceedings.

1. The Radiative Transfer Equations

The polarization properties of light are described by the four Stokes parameters I, Q, V, and U. I is the intensity. Linear polarization is described by Q and U, and circular polarization by V. In general, the absorption coefficients κ_l and κ_r for left- and right-handed circularly polarized light, and κ_p for linearly polarized radiation traveling perpendicular to the magnetic field, are not equal at a given wavelength. κ_l, κ_p, and κ_r correspond to transitions where the magnetic quantum number changes by $\Delta m = -1, 0,$ and $+1$. The absorption coefficients, normalized to the Rosseland mean opacity ($\kappa_{\rm Ross}$), are defined by $\eta_p = \kappa_p/\kappa_{\rm Ross}$, $\eta_l = \kappa_l/\kappa_{\rm Ross}$, and $\eta_r = \kappa_r/\kappa_{\rm Ross}$. These absorption coefficients are combined into $\eta_I = \frac{1}{2}\eta_p \sin^2\psi + \frac{1}{4}(\eta_l + \eta_r)(1 + \cos^2\psi)$, $\eta_Q = \frac{1}{2}\eta_p - \frac{1}{4}(\eta_l + \eta_r)\sin^2\psi$, and $\eta_V = \frac{1}{2}(\eta_r - \eta_l)\cos\psi$. Here ψ denotes the angle between the magnetic field direction and the line of sight. When the magneto-optical parameters ρ_R (Faraday rotation) and ρ_W (Voigt effect) originating from spectral lines and free electrons in a magnetic field (see Jordan et al. 1992) are taken into account, the three radiative transport equations of Unno (1956) expand into four equations (Beckers 1969):

$$\mu\frac{\mathrm{d}I}{\mathrm{d}\tau} = \eta_I(I - B) + \eta_Q Q + \eta_V V, \tag{1}$$

$$\mu\frac{\mathrm{d}Q}{\mathrm{d}\tau} = \eta_Q(I - B) + \eta_I Q + \rho_R U, \tag{2}$$

$$\mu\frac{\mathrm{d}U}{\mathrm{d}\tau} = \rho_R Q + \eta_I U - \rho_W V, \tag{3}$$

$$\mu\frac{\mathrm{d}V}{\mathrm{d}\tau} = \eta_V(I - B) + \rho_W U + \eta_V V. \tag{4}$$

The optical depth scale is $d\tau \equiv -\kappa_{\mathrm{Ross}} dz'$, and $\mu \equiv \cos\vartheta$, where ϑ is the angle between the normal to the surface and the line of sight (along which the z' axis of the local coordinate system on each surface element points). The x' axis is defined by the direction of the projection of the magnetic field vector onto the plane of the sky. $y\prime$ completes the right-handed coordinate system. B is the source function (the Planck formula in the case of LTE).

In the case of large Faraday rotation, radiative transfer can be described in the framework of generalized Stokes parameters, with I_+ being the ordinary mode and I_- being the extraordinary mode. For the ordinary mode the electric field vector, the magnetic field, and the direction of propagation are in one plane; for the extraordinary mode the plane of propagation and the electric field vector is perpendicular to the magnetic field. The phase relation between the two modes is given by $I_c = \sqrt{I_+ I_-}\cos\delta$, and $I_s = \sqrt{I_+ I_-}\sin\delta$, which correspond to two polarization ellipses, the so-called normal modes.

We have used three different methods to solve the radiative transfer equations formally: (a) The slightly modified Martin und Wickramasinghe (M&W, 1979), (b) an accelerated lambda iteration scheme (ALI), (c) an approximation for large Faraday rotation (APPROX), or (d) an analytical procedure using matrix exponential function (MATEXP).

1.1. Algorithm by M&W

The solution is based on the assumption that the source function is linear in the optical depth and that between two successive depth points the Stokes parameters can be described by

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I_a \\ Q_a \\ U_a \\ V_a \end{pmatrix} + \begin{pmatrix} I_b \\ Q_b \\ U_b \\ V_b \end{pmatrix}\tau + \sum_{i=4}^{4} \begin{pmatrix} I_{ci} \\ Q_{ci} \\ U_{ci} \\ V_{ci} \end{pmatrix} \exp(a_i\tau). \tag{5}$$

This assumption is inserted into the four radiative transfer equations, using an Unno (1956) condition at the inner boundary. Then the coefficients $I_a, \dots V_{ci}$, as well as a_i, are evaluated by comparing the constant, linear, and exponential terms on both sides of the equations. One obtains finally a relatively simple scheme, by which the transport equations can be solved from the inner depth points to the outside.

1.2. ALI method

This powerful and very flexible method is described in more detail by Deetjen et al. in these proceedings.

1.3. APPROX method

This method uses generalized Stokes parameters (Ramaty 1969). Of particular interest is the limiting case of large Faraday rotation, which is always valid in the case of magnetic white dwarfs. This means that the rotation between two depth points is larger than 2π, and the phase relation between the polarization ellipses I_s and I_c can assume arbitrary values; $I_s = I_c = 0$ on the average. Then the transfer equations decouple (Ramaty 1969):

$$\mu \frac{dI_\pm}{d\tau} = \alpha_\pm \left(\frac{B}{2} - I_\pm \right). \tag{6}$$

According to Väth & Chanmugam (1995) the formal solution (using a linear approximation of the source function) of this equation can be written as

$$\begin{aligned} I_+(\tau_i) &= \frac{1}{2}\alpha_B(1 - e^{-\alpha_+\Delta\tau/\mu})\frac{1}{2}\frac{\beta_B}{\alpha_+}\left[\alpha_+(\tau_i - \tau_{i+1}e^{-\alpha_+\Delta\tau/\mu}) - (1 - e^{-\alpha_+\Delta\tau/\mu})\right] \\ &\quad + I_+(\tau_{i\pm1})e^{-\alpha_+\Delta\tau/\mu} \\ I_-(\tau_i) &= \frac{1}{2}\alpha_B(1 - e^{-\alpha_-\Delta\tau/\mu})\frac{1}{2}\frac{\beta_B}{\alpha_-}\left[\alpha_+(\tau_i - \tau_{i+1}e^{-\alpha_-\Delta\tau/\mu}) - (1 - e^{-\alpha_-\Delta\tau/\mu})\right] \\ &\quad + I_-(\tau_{i\pm1})e^{-\alpha_-\Delta\tau/\mu} \end{aligned}$$

1.4. MATEXP method

A direct analytical way to solve the radiative transfer equations uses matrix exponential functions. After Dittmann (1995) the radiative transfer equation $d\vec{I}/d\tau = \boldsymbol{\eta}(\vec{I} - \vec{S})$, where $\vec{I} \equiv (I, Q, V, U)$, can be written as

$$\vec{I}(0) = \exp(-\boldsymbol{\eta}\tau)\left[\vec{I}(\tau) - \boldsymbol{\eta}\int_\tau^0 \exp(-\boldsymbol{\eta}s)\,\vec{S}(s)\,ds\right] \tag{7}$$

where the matrix exponential function is defined as $e^{\boldsymbol{\eta}} \equiv \sum_{n=0}^{\infty} \boldsymbol{\eta}^n/n!$. The infinite series can be rewritten as $e^{\boldsymbol{\eta}\tau} = \sum\limits_{i=0}^{3} c_i(\tau)\mathbf{M}^i$. The matrix $\mathbf{M}$ is given by $\mathbf{M} = \boldsymbol{\eta} - \eta_I\mathbf{E}$. We assume a linear source function between successive depth points: $S = (\alpha + \beta\tau, 0, 0, 0)$. If we define $x = \frac{1}{2}\sqrt{2(\eta_5^2 + \rho_3^2)}$, $y = \frac{1}{2}\sqrt{2(\eta_5^2 - \rho_3^2)}$, $z = -iy$, $\rho_3^2 = \sqrt{\eta_5^4 + 4(\eta_Q\rho_W + \eta_V\rho_R)^2}$, $\eta_5 = \sqrt{\eta_Q^2 + \eta_V^2 - \rho_R^2 - \rho_W^2}$ and take into account that y can be become imaginary if magneto-optical parameters are present (a case that has been neglected by Dittmann 1995), then the coefficients are given by:

$$c_0 = \frac{e^{\eta_I \tau}}{\rho_3^2}\left[x^2\cos(z\tau) - y^2\cosh(x\tau)\right] \quad (8)$$

$$c_1 = \frac{e^{\eta_I \tau}}{\rho_3^2}\left[\frac{x^2}{z}\sin(z\tau) - \frac{y^2}{x}\sinh(x\tau)\right] \quad (9)$$

$$c_2 = \frac{e^{\eta_I \tau}}{\rho_3^2}\left[\cosh(x\tau) - \cos(z\tau)\right] \quad (10)$$

$$c_3 = \frac{e^{\eta_I \tau}}{\rho_3^2}\left[\frac{1}{x}\sinh(x\tau) - \frac{1}{z}\sin(z\tau)\right] \quad (11)$$

For large magneto-optical parameters $\lim_{x\to 0}\sinh(x\tau)/x = \tau$, and the expressions can be somewhat simplified.

2. Conclusion

In all tests, the resulting spectra and polarization turned out to be very similar for the four different approaches. Small numerical instabilities in the case of the M&W and the MATEXP algorithms lead to small oscillations of the Stokes parameters (particularly Q and V) in the vicinity of spectral lines; thsese are caused by the terms containing sin, cos, sinh, or cosh. The ALI method, having the potential to account for NLTE and scattering is numerically very stable. However, the APPROX method is about a factor of 11 faster and very stable and should therefore be preferred for practical purposes. For final fits to observations it is always possible to use the ALI method for comparison.

References

Beckers, J. M. 1969, Sol.Phys., 9, 372

Dittmann, O. 1995, PhD thesis, Heidelberg

Jordan S., 1992, A&A, 265, 570

Jordan, S. 2001, in ASP Conf. Ser. Vol. 226 (San Francisco: ASP), 269

Martin, B., Wickramasinghe D., 1979, MNRAS, 189, 883

Ramaty, R. 1969, ApJ, 158, 753

Unno, W. 1956, PASJ, 8, 108

Väth, H., Chanmugam, G. 1995, ApJS, 98, 295

Wickramasinghe, D. T. & Ferrario, L. 2000, PASP, 112, 873